

A note on estimating stochastic volatility and its volatility: a new simple method

With the exception of Alghalith (2012), the previous methods of estimating (stochastic) volatility suffered two major limitations. Firstly, they are only applicable to *time-series* data and therefore the volatility of the investor's portfolio (a cross section of assets) for time t cannot be properly and directly estimated. Secondly, these methods required the arbitrary generation of data series for volatility in order to estimate the volatility. This seems somewhat contradictory and futile. Alghalith (2012) used a stochastic factor model, where volatility is a function of an external economic factor (GDP). In this note, we use a *stochastic* volatility (in the sense of Heston, etc.). In doing so, we overcome the aforementioned limitations. Below is a description of the theoretical model.

We modify the standard portfolio model to include a stochastic volatility. We still use a two-dimensional Brownian motion $\{W_{1s}, W_{2s}, \mathcal{F}_s\}_{t \leq s \leq T}$ on the probability space $(\Omega, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t \leq s \leq T}$ is the augmentation of filtration. We include a risky asset and a risk-free asset. The risk-free asset price process is given by $S_0 = e^{\int_t^T r ds}$, where $r \in C_b^2(R)$ is the rate of return.

The dynamics of the risky asset price are given by

$$dS_s = S_s \{ \mu_s ds + \sigma_s dW_{1s} \}, \quad (1)$$

where μ_s and σ_s are the rate of return and the volatility, respectively.

As in Heston's model, the stochastic volatility is given by

$$d\sigma_s^2 = (\alpha - \beta\sigma_s^2) ds + \gamma\sigma_s dW_{2s}, \sigma_t^2 \equiv \bar{\sigma}, \quad (2)$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions,

α , β and γ are constants.

The wealth process is given by

$$X_T^\pi = x + \int_t^T \{ r_s X_s^\pi + (\mu_s - r_s) \pi_s \} ds + \int_t^T \pi_s \sigma_s dW_{1s}, \quad (3)$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t \leq s \leq T}$ is the portfolio process, and

$E \int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, \bar{\sigma})$ is admissible. $X_s = \pi_s + B_s$, where B_s is the amount invested in the risk-free asset.

The investor's objective is to maximize the expected utility of the terminal wealth

$$V(t, x, \bar{\sigma}) = \sup_{\pi} E[U(X_T^{\pi}) | \mathcal{F}_t],$$

where $V(\cdot)$ is the value function, $U(\cdot)$ is continuous, smooth, bounded and strictly concave utility function.

The value function satisfies the Hamilton-Jacobi-Bellman PDE (suppressing the notations)

$$\begin{aligned} & V_t + rxV_x + (\alpha - \beta\bar{\sigma})V_{\bar{\sigma}} + \frac{1}{2}\gamma^2\bar{\sigma}V_{\bar{\sigma}\bar{\sigma}} + \\ & \sup_{\pi} \left\{ \frac{1}{2}\pi^2\bar{\sigma}V_{xx} + [\pi(\mu - r)]V_x + \gamma\rho\bar{\sigma}\pi V_{x\bar{\sigma}} \right\} = 0, \\ & V(T, x, \bar{\sigma}) = u(x, \bar{\sigma}). \end{aligned}$$

Thus the solution yields

$$\pi^* = -\frac{(\mu - r)V_x}{\bar{\sigma}V_{xx}} - \rho\gamma\frac{V_{x\bar{\sigma}}}{V_{xx}}. \quad (4)$$

We take a second-order Taylor's expansion of $V(t, x, \bar{\sigma})$ and therefore

$$V_x(t, x, \bar{\sigma}) \approx \alpha_0 + \alpha_1 x + \alpha_2 \bar{\sigma}.$$

Substituting this into (4) yields

$$\pi^* = -\frac{(\mu - r)[\alpha_0 + \alpha_1(\pi^* + B) + \alpha_2\bar{\sigma}]}{\alpha_1\bar{\sigma}} - \rho\gamma\frac{\alpha_2}{\alpha_1}.$$

The above equation can be rewritten as

$$\pi^* = \beta_2 - \frac{(\mu - r)\pi^*}{\bar{\sigma}} + \frac{\beta_1(\mu - r)}{\bar{\sigma}}; \beta_1 \equiv -\frac{\alpha_0 + \alpha_1 B + \alpha_2\bar{\sigma}}{\alpha_1}, \beta_2 \equiv -\rho\gamma\frac{\alpha_2}{\alpha_1}, \quad (5)$$

where β_i is a constant. Thus

$$\pi^* = \frac{\beta_2}{1 + \frac{\mu - r}{\bar{\sigma}}} + \frac{\beta_1(\mu - r)}{\bar{\sigma} + \mu - r}.$$

The above equation can be rewritten as the following regression equations

$$\pi_t^* = \frac{\beta_2}{1 + \frac{\mu - r}{\beta_3}} + \frac{\beta_1(\mu - r)}{\beta_3 + \mu - r}, \quad (6)$$

where β_i is a parameter to be estimated by a non-linear regression (while the variables π^* , μ and r are observed data), and β_3 is an estimate of the volatility of the portfolio for period t .

To estimate the volatility of volatility γ , we multiply (6) by γ and take

the inverse to obtain

$$\breve{\pi} = \frac{\beta_4}{\frac{\beta_5}{1+\frac{\mu-r}{\beta_3}} + \frac{\beta_6(\mu-r)}{\beta_3+\mu-r}}, \beta_4 \equiv \gamma, \breve{\pi} \equiv \frac{1}{\pi_t^*},$$

where β_4 is an estimate of γ , $\hat{\beta}_3$ is the estimated volatility from the previous regression, and $\breve{\pi}$ is observed data. A similar procedure can be used to estimate the factor of correlation.

References

- [1] Alghalith (2012). New methods of estimating volatility and returns: Revisited. *Journal of Asset Management*, 13, 307–309.